Complexity of Local Search for Euclidean Clustering Problems

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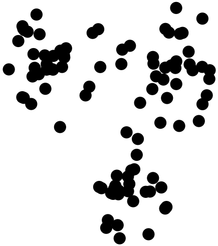
Clustering

Given n points in $\mathbb{R}^d$, group the points into <u>clusters</u>.



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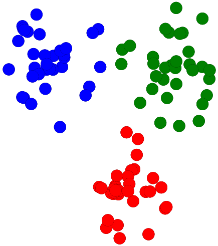

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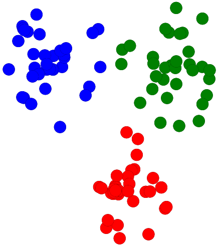

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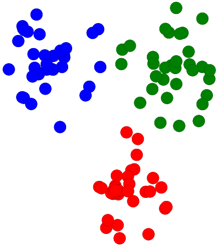

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- *k*-Means: minimize $\sum_{i=1}^{k} \sum_{x \in C_i} \|x - \mathrm{cm}(C_i)\|^2$.

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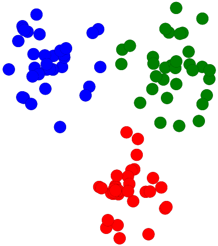

Different objectives:

- k-Means: minimize $\sum_{i=1}^{k} \sum_{x \in C_i} \|x - \mathrm{cm}(C_i)\|^2$.
- Squared Euclidean Max Cut: maximize $\sum_{x \in X} \sum_{y \in Y} \|x - y\|^2$.

Flip Heuristics for Clustering

Simple local search method: reassign single points.

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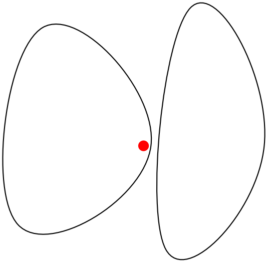

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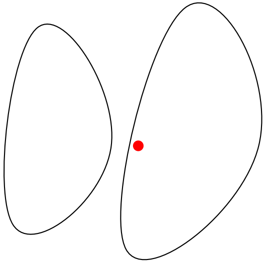

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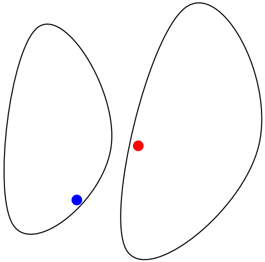

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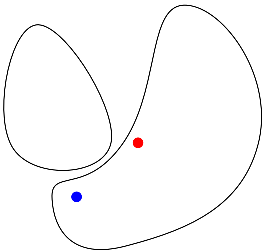

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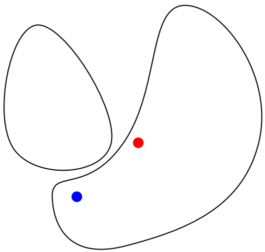

Theorem (Etscheid & Röglin, Manthey & R)

There exist instances of both k-Means and Squared Euclidean Max Cut that require $2^{\Omega(n)}$ iterations of Hartigan–Wong and Flip, respectively.

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Need a notion of reduction between PLS problems.

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Crucial: if s' is locally optimal, then so is $s = g(s')$.

Implications

P is PLS-complete $\implies$ P among hardest problems in PLS.

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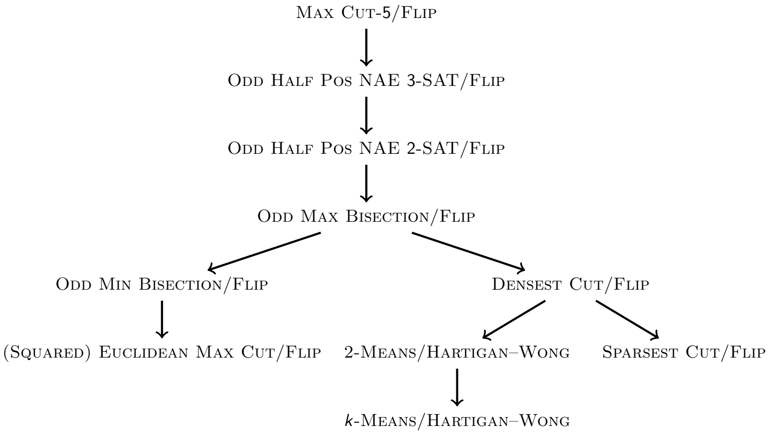
Theorem (Schäffer & Yannakakis)

If for some $P, Q \in$ PLS we have $P \leq_{PLS} Q$ via a tight reduction, then Q inherits any lower bounds on the worst-case running time of P.

Reduction Path



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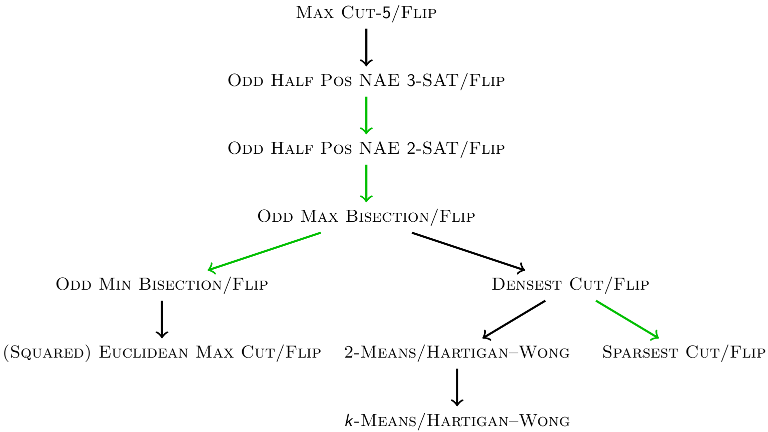

Reduction Path

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$\downarrow$

Odd Half Pos NAE 3-SAT/Flip

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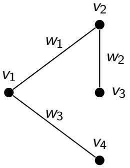

$$\mathcal{X} = \begin{pmatrix} \sqrt{w_1/2} & 0 & \sqrt{w_3/2} & \alpha_{v_1} & 0 & 0 & 0 \\ \sqrt{w_1/2} & \sqrt{w_2/2} & 0 & 0 & \alpha_{v_2} & 0 & 0 \\ 0 & \sqrt{w_2/2} & 0 & 0 & 0 & \alpha_{v_3} & 0 \\ 0 & 0 & \sqrt{w_3/2} & 0 & 0 & 0 & \alpha_{v_4} \end{pmatrix}$$

with $\alpha_v = \sqrt{w(E)/2 - w(\delta(v))/2}$

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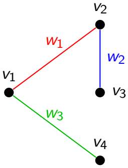

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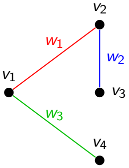

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Now we are in a purely combinatorial setting $\rightarrow$ more freedom.

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Goal: maximize weight of satisfied clauses.

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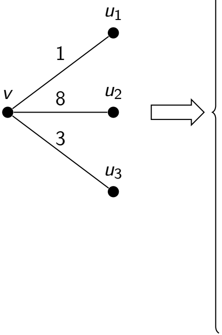

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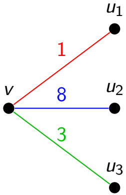

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Level 1

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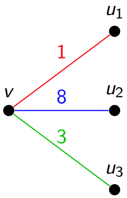

$\Rightarrow$

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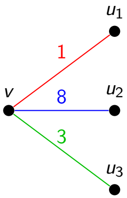

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$\mathrm{NAE}(v, q_v, a_i)$	$\{u_1\}$	weight: 0	
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$\mathrm{NAE}(v, q_v, a_i)$	$\emptyset$	weight: 0	

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Other PLS-complete Euclidean optimization problems, e.g. TSP/k-Opt?

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